Order isomorphisms of complete order-unit spaces

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Abstract. We investigate order isomorphisms, which are not assumed to be linear, between complete order unit spaces. We show that two such spaces are order isomorphic if and only if they are linearly order isomorphic. We then introduce a condition which determines whether all order isomorphisms on a complete order unit space are automatically affine. This characterisation is in terms of the geometry of the state space. We consider how this condition applies to several examples, including the space of bounded self-adjoint operators on a Hilbert space. Our techniques also allow us to show that in a unital C^*-algebra there is an order isomorphism between the space of self-adjoint elements and the cone of positive invertible elements if and only if the algebra is commutative.

1. Introduction

A basic problem concerning ordered vector spaces is to characterise their order isomorphisms. These are the bijective maps that preserve the order in both directions. In many cases, these maps are necessarily affine, and we would like to know under which conditions this is so.

The earliest rigidity results on order isomorphisms of ordered vector spaces were by Alexandrov [1, 2] and Zeeman [27]. Their motivation came from Special Relativity, and so they were interested in the Lorentz cone, although Alexandrov later broadened his result to general finite-dimensional cones. This work was extended in [22, 20, 21, 6, 13, 24]. The techniques used in these papers rely on the existence of extreme rays of the cone defining the order, and work well when there are many such rays. However, many interesting cones have few or no extreme rays, and here the techniques break down.

The approach taken in this paper is to study the dual space and the dual cone. It is convenient to reduce the generality slightly by assuming that the vector space is ordered by an Archimedean cone and has an order unit. The advantage of this is that under these assumptions the dual cone has a cross section, called the state space, which is compact in the weak* topology. By the Krein Milman theorem, therefore, the state space has extreme points, the pure states. We call the closure of the set of pure states the *pure state space*. One can represent points of the original vector space V as continuous functions on the pure state space, satisfying certain constraints. These constraints are given by the *affine dependencies*. These are the non-zero signed measures μ supported on the pure state space such that

$$\int_K g(x)\,\mathrm{d}\mu(x) = 0, \qquad \text{for all } g \in V.$$

Our first theorem shows that the existence of an order isomorphism between two complete order unit spaces implies that they are linearly isomorphic. Actually, the result

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holds even when the map is only defined on certain subsets. We say that a subset X of a partially ordered space is *upper* if x is in X whenever $y \in X$ and $y \leq x$.

Theorem 1.1. *Let (V, C, u) and (V', C', u') be two order unit spaces, each of which is complete under its respective order unit norm. If there is an order isomorphism between upper sets $X \subset V$ and $X' \subset V'$, then there is a linear order-isomorphism between V and V'.*

The following example shows that some assumptions on the ordered vector spaces are necessary for this result to hold. Let (Ω, Σ, μ) be a measure space, and take $p \in (1, \infty)$. The map $f \mapsto |f|^p \operatorname{sign} f$ is an order isomorphism between the spaces $L^p(\Omega, \Sigma, \mu)$ and $L^1(\Omega, \Sigma, \mu)$. Here $\operatorname{sign} x$ denotes the sign of x. However, there is no linear order isomorphism between these two spaces if the dimension is infinite. Note that, in this case, neither of the spaces is an order unit space. This example is contained in [14]. In the same paper, it was shown that the conclusion of Theorem 1.1 is true for certain Banach lattices, without assuming that there is an order unit.

Our next theorem concerns the question of when all order isomorphisms between complete order unit spaces are affine. For some spaces this is not the case; for example, there are many non-affine order isomorphisms from the real line with its usual order to itself. This example generalises to the product of the real line with any ordered vector space—take a non-affine order isomorphism on the real line and the identity on the complement. We introduce the following condition.

Condition 1.2. *Every pure state is contained in the closure of the union of the supports of the affine dependencies.*

A set X of a partially order space is said to be *directed downward* if, for every x and y in X, there exists z in X such that $z \leq x$ and $z \leq y$. Note that an order unit space is itself directed downward, as is its closed cone and its open cone.

Theorem 1.3. *Let V and V' be two complete order-unit spaces, containing non-empty upper sets U and U' respectively. If V satisfies Condition 1.2, then every order isomorphism between U and U' is affine. Conversely, if there exists an order isomorphism between U and U', every such isomorphism is affine, and U is directed downward, then Condition 1.2 holds.*

The additional assumption of downward directedness is necessary in the converse direction because sometimes all order isomorphisms may be affine for reasons more to do with the structure of the domain than the geometry of the space. An example of this is the half space $\{(x, y, z) \mid x + y + z \geq 0\}$ in $\mathbb{R}^3$ with the product order. One can show that $\mathbb{R}^3$ does not satisfy Condition 1.2, but yet all order isomorphisms from the half-space to itself are affine.

Other situations where the domain affects which order isomorphisms exist have been studied in [25] and [17].

From Theorem 1.3 we may recover a known result about the space $B(\mathcal{H})_{sa}$ of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, ordered by the cone of those that are positive semi-definite.

Corollary 1.4. *Let $\mathcal{H}$ be a Hilbert space of dimension at least two, and let U, $U' \subset B(\mathcal{H})_{sa}$ be upper sets. Then every order-isomorphism $\varphi\colon U \to U'$ is affine.*

This result was proved in [15] when the sets are either the closed cone or whole space, and more generally in [13]. In understanding how it follows from Theorem 1.3,

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The situation where there are no affine dependencies at all supported on the pure state space occurs precisely when the state space is a *Bauer simplex* (see [3, Theorem II.4.1]). An equivalent condition is that the ordered vector space is of the form $C(K)$, the space of continuous real-valued functions on a compact Hausdorff space K. Our techniques allow us to prove the following.

Theorem 1.5. *Let (V, C, u) be an order unit space that is complete under its order unit norm. There is an order isomorphism between $\operatorname{int} C$ and V if and only if the state space of V is a Bauer simplex.*

A corollary of this result was conjectured by Molnár in [16].

Corollary 1.6. *Let $\mathcal{A}$ be a unital C^*-algebra. Then, there is an order isomorphism between the cone $\mathcal{A}_+^{-1}$ of invertible positive elements and the space $\mathcal{A}_{\mathrm{sa}}$ of self-adjoint elements if and only if $\mathcal{A}$ is commutative.*

Proof. We use that a unital C^*-algebra is commutative if and only if its state space is a Bauer simplex; see Corollary 2.4 of [7]. □

In [16], Molnár settled the case of the algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$, and also of the finite dimensional non-commutative C^*-algebras.

The content of the present paper is as follows. After recalling some background material in section 2, we prove the easier direction of Theorem 1.3 in section 3. In section 4, we study the order ideals of the cone of an order unit space. We use a purely order theoretic notion of ideal, which is appropriate here because we are interested in maps that are not assumed to preserve any linear structure. In section 5, we examine how these ideals behave under order isomorphisms. The main outcome is that there is a homeomorphism induced between the pure state spaces of the domain and the image. This generalises Kaplansky's theorem [11] concerning the space $C(K)$ of continuous real-valued functions on a compact space K. In a follow-up paper [12], Kaplansky determined the form of the order isomorphisms on $C(K)$ that are continuous in the supremum norm topology. He showed that associated to each pure state is a function that determines how that coordinate is transformed. We find that a similar expression holds in our setting; see Proposition 5.6. In section 6, we restrict our attention to order isomorphisms on a cone where the induced homeomorphism is the identity map, and show that, for each pure state in the support of an affine dependency, the transformation map is affine. We use this in section 7 to show that in the general case the coordinate transformation associated to a pure state in the support of an affine dependency is homogeneous of some degree. In section 8, we show that an order isomorphism all of whose coordinate transformations are homogeneous is Gateaux differentiable, and we prove Theorem 1.1. We complete the proofs of Theorems 1.3 and 1.5 in section 9. Finally in section 10, we provide some examples, including a proof of Corollary 1.4.

2. Preliminaries

A useful reference for this section is Alfsen's book [3].

2.1. **Order unit spaces.** Let V be a real vector space. A cone C in V is a subset that is closed under addition and under multiplication by non-negative real numbers, and satisfies $C \cap -C = \{0\}$. Such a cone induces a partial order $\leq$ on V when we set $x \leq y$

whenever $y - x \in C$. An ordered vector space is a vector space with a partial order determined in this way from a cone.

An ordered vector space V is said to be *Archimedean* if $x \leq 0$ whenever $x \in V$ and $y \in C$ satisfy $nx \leq y$, for all $n \in \mathbb{N}$. In finite dimension, this is equivalent to the cone being closed. An *order unit* is an element u of the cone such that for each $x \in V$ there is a $\lambda > 0$ such that $x \leq \lambda u$. When V is equipped with an Archimedean cone C and an order unit u, it becomes an *order-unit space* (V, C, u). The following norm is called the *order-unit norm*:

$$||x||_u := \inf\{\lambda > 0 \mid -\lambda u \leq x \leq \lambda u\}, \qquad \text{for all } x \in V.$$

When V has the topology induced by this norm, the cone C is closed and has non-empty interior, which we denote by $\operatorname{int} C$. Indeed, the interior is exactly the set of order units of V. By a *complete* order unit space, we mean one that is complete with respect to its order-unit norm.

Let V and V' be two ordered vector spaces. A map φ from a subset X of V to a subset X' of V' is said to be *isotone* or *order-preserving* if $x \leq y$, for two elements x and y of X, implies that $\varphi(x) \leq \varphi(y)$. If φ is bijective and both it and its inverse are order preserving, then φ is said to be an *order isomorphism* between X and X'.

Lemma 2.1. *Let $\varphi \colon C \to C'$ be an order isomorphism between the cones of two order-unit spaces. Then, there exist order units u and u' in C and C', respectively, such that $\varphi(u) = u'$.*

Proof. Let v and v' be order units of C and C', respectively. Choose $\lambda > 0$ such that $u := \lambda v \geq \varphi^{-1}(v')$. Observe that u is an order unit of C. Since $u' := \varphi(u) \geq v'$, we also have that u' is an order unit of C'. □

2.2. **The dual space.** Let (V, C, u) be an order-unit space endowed with the topology coming from its order-unit norm, and denote by V^* the topological dual space. In other words, V^* is the space of continuous real-valued functionals on V. The weak* topology on V^* is coarsest topology such that each element of V is a continuous function. It is characterised by the following convergence criterion: a net y_α in V^* converges to an element y of V^* if and only if $y_\alpha(x)$ converges to $y(x)$ for all x in V. The dual cone C^* of C is the subset of V^* consisting of positive functionals, that is,

$$C^* := \big\{y \in V^* \mid y(x) \geq 0, \text{ for all } x \in C\big\}.$$

The cross-section

$$K := \big\{y \in C^* \mid y(u) = 1\big\}$$

is compact in the weak* topology. The elements of this set are called the *states* of V. The extreme points of K are called the *pure states*, and they play an important role. An extreme point of K is an element e such that there is no line segment contained in K with e in its relative interior. By the Krein–Milman theorem, K is guaranteed to have extreme points—indeed, it is the closed convex hull of the set of these points. We denote by $\partial_e K$ the set of pure states. The closure of the set of pure states is called the *pure state space*, and we denote it $F := \operatorname{cl} \partial_e K$. Here cl denotes the topological closure of a set. We have that $x \in V$ is in C if and only if $p(x) \geq 0$, for all states p.

2.3. **Choquet representation.** We denote by $A(K)$ the set of affine real-valued functions on K that are continuous in the weak* topology. Every element of $A(K)$ can be extended to a continuous linear functional on whole of V^* if and only if V is complete in the order-unit norm; see [3, Theorem II.1.8].

The support $\operatorname{supp}\mu$ of a measure μ on a measurable topological space Ω is the largest subset of Ω having the property that every open neighbourhood of each of its points has positive measure. The support of a measure is always closed. The support of a signed measure μ is defined to be $\operatorname{supp}\mu := \operatorname{supp}\mu^+ \cup \operatorname{supp}\mu^-$, where $\mu = \mu^+ - \mu^-$ is the Hahn decomposition.

A measure μ on K is said to *represent* a point $y \in K$ if the barycenter formula holds for all $g \in A(K)$, that is,

$$g(y) = \int_K g(x)\,\mathrm{d}\mu(x), \qquad \text{for all } g \in A(K).$$

By taking g to be the constant function 1, we see that any measure representing a point must be a probability measure, that is, have total mass 1. The Choquet–Bishop–de Leeuw theorem states that every point in K can be represented by a probability boundary measure, that is, a measure of total mass 1 that is maximal with respect to a certain partial order on the set of measures on K. The details of this partial order are not important to us—we will only need that the support of any boundary measure is a subset of the pure state space $F := \operatorname{cl}\partial_e K$; see [3, Proposition I.4.6].

2.4. **Affine dependencies.** A non-zero signed measure μ on K said to be an *affine dependency* if

$$\mu(g) := \int_K g(x)\,\mathrm{d}\mu(x) = 0, \qquad \text{for all } g \in A(K).$$

Observe that, if μ_1 and μ_2 are probability measures supported on K representing a point $y \in K$, then $\mu_1 - \mu_2$ is an affine dependency. In fact, by considering the Hahn decomposition of an affine dependency, it is easy to see that all affine dependencies arise in this way, up to a scalar factor.

The following lemma establishes a criterion for when a function defined on F is the restriction of an element of V. This result is known (see the remark on page 108 of [3]), but we give an explicit proof for convenience.

Lemma 2.2. *Assume that V is a complete order-unit space. Let $f\colon F \to \mathbb{R}$ be a continuous function such that $\mu(f) = 0$ for every affine dependency supported by F. Then, there exists an element g of V such that $g(p) = f(p)$ for all $p \in F$.*

Proof. Define

$$g(x) := \int_F f(\xi)\,\mathrm{d}\mu_x(\xi), \qquad \text{for all } x \in K.$$

Here μ_x is any probability measure on F representing the point x.

First we show that the choice of representing measure is not important. Suppose that $x \in K$ can be represented by two different measures μ_x and μ'_x on F. Observe that $\mu_x - \mu'_x$ is an affine dependency, and so $(\mu_x - \mu'_x)(f) = 0$. Therefore,

$$\int_F f(\xi)\,\mathrm{d}\mu_x(\xi) - \int_F f(\xi)\,\mathrm{d}\mu'_x(\xi) = \int_F f(\xi)\,\mathrm{d}(\mu_x - \mu'_x)(\xi) = 0.$$

Thus, it does not matter whether we integrate with respect to μ_x or μ'_x.

We wish to show that g is continuous. Let x_α be a net in K converging to a limit $x \in K$. By taking a subnet if necessary, we may assume that $g(x_\alpha)$ converges to some limit. For each α, choose a probability measure μ_{x_α} on F representing x_α. Since F is compact, there is a subnet μ_{x_β} of this net of measures converging to a limiting probability measure μ. By evaluating along the net x_β an arbitrary continuous affine function on K, we see that μ represents x. The map f is continuous, and so we deduce that $g(x_\alpha)$ converges to $g(x)$.

To show that g is affine, let x_1 and x_2 be two points in K, represented, respectively, by probability measures μ_{x_1} and μ_{x_2} on F. Each convex combination $(1-\lambda)x_1 + \lambda x_2$, with $\lambda \in [0,1]$, of these two points can be represented by the same convex combination $(1-\lambda)\mu_{x_1} + \lambda\mu_{x_2}$ of the measures. Since integrals are linear in the measure, we see that g is an affine function on K.

We have shown that g is well-defined, continuous, and affine; hence it is the restriction to K of an element of V. Here we are using that V is complete in its order-unit norm. □

2.5. **Extension of positive maps.** For a proof of the following proposition, see for example [5, Lemma 1.26]. Note that the assumption in that version that C' is Archimedean is not needed here because we are assuming positive homogeneity.

Proposition 2.3. *Let (V, C) and (V', C') be two ordered vector spaces, and let $\varphi\colon C \to C'$ be an additive and positively homogeneous map between their cones. Assume that C generates V. Then, φ can be extended in a unique way to an order-preserving linear map from V to V'.*

2.6. **The Pettis integral.** Let $f\colon X \to V$ be a function from a measure space (X, Σ, μ) to a topological vector space V admitting a dual space V^* that separates points. We say that f is *Pettis integrable* if $y \circ f$ is Lebesgue integrable for each $y \in V^*$ and there exists an element v of V satisfying

$$\langle y, v\rangle = \int_X \langle y, f(x)\rangle \, \mathrm{d}\mu(x), \qquad \text{for all } y \in V^*.$$

The point v is called the *Pettis integral* of f. See [18] for more information about this integral.

3. The existence of non-linear order isomorphisms

In this section, we show that non-affine order isomorphisms exist when Condition 1.2 is not satisfied. Throughout the section, we assume that V is an order unit space that is complete under its order unit norm. Recall that F is the weak* closure of the set of extreme points $\partial_e K$ of the state space K.

Let L be the weak* closure of the union of the supports of the affine dependencies supported by F, that is,

$$L := \mathrm{cl} \bigcup \big\{ \operatorname{supp} \mu \mid \mu \text{ is an affine dependency supported by } F \big\}. \tag{1}$$

Clearly, L is a subset of F. Condition 1.2 is that the two sets are equal.

A Tychonoff space is a topological space in which each closed set can be separated by a continuous real-valued function from any point that it does not contain. More precisely, in a Tychonoff space, if A is a closed set, and x is a point not in A, then there exists a real-valued continuous function on the space that takes the value 0 on all of A, and the value 1 at x. Tychonoff spaces are also called "completely regular" or $T_{3\frac{1}{2}}$. A compact Hausdorff space is always Tychonoff.

Lemma 3.1. *Let X be a non-empty subset of V that is upper and directed downward. If Condition 1.2 does not hold, that is, if $L \neq F$, then there exists an order isomorphism from X to itself that is not affine.*

Proof. By conjugating with translations if necessary, we may assume without loss of generality that 0 is an element of X.

Let $w \in F$ lie outside L. Since F is a compact Hausdorff space, it is Tychonoff. So, there exists a continuous function $s\colon F \to \mathbb{R}$ that separates w from L; in fact, we may choose s to take the value 0 on all of L, the value 1 at w, and values in $[0,1]$ everywhere else.

Choose a continuous function $\Gamma\colon [0,1] \times [-\infty,\infty] \to [-\infty,\infty]$ with the following properties. We require that, for each $y \in [0,1]$, the map $\Gamma(y,\cdot)$ is an order isomorphism of $[-\infty,\infty]$ and agrees with the identity map outside the interval $[1,2]$. We furthermore require that $\Gamma(0,\cdot)$ is the identity map, and that $\Gamma(1,\cdot)$ is non-affine in the interval $[1,2]$.

Let g be in V. The map $f\colon F \to \mathbb{R}$ defined by $f(p) := \Gamma(s(p), g(p))$ is continuous. Let μ be an affine dependency supported by F. The support of μ is a subset of L, and on this set the function s takes the value 0. Using then that $\Gamma(0,\cdot)$ is the identity map, we get that $\mu(f) = \mu(g) = 0$. Applying Lemma 2.2, we see that there is an element of V agreeing with f on F. We denote this element φg. This defines a map φ from V to itself.

For each $y \in [0,1]$, define the function $\bar{\Gamma}(y,\cdot)$ to be the inverse of $\Gamma(y,\cdot)$. Let y_n be a sequence in $[0,1]$ converging to y in the same set, and let x_n be a sequence in $[-\infty,\infty]$ converging to $x \in [-\infty,\infty]$. Writing $z_n := \bar{\Gamma}(y_n, x_n)$, we have $x_n = \Gamma(y_n, z_n)$, for all $n \in \mathbb{N}$. If any subsequence of z_n converges to a limit $z \in [-\infty,\infty]$, then $x = \Gamma(y,z)$, which is equivalent to $z = \bar{\Gamma}(y,x)$. We conclude that z_n converges to $\bar{\Gamma}(y,x)$, which shows that $\bar{\Gamma}$ is continuous. Note that $\bar{\Gamma}$ also satisfies all the other assumptions we made on Γ.

Analogously to how we defined φ, we can define a map $\bar{\varphi}\colon V \to V$ such that

$$\bar{\varphi} g(p) = \bar{\Gamma}\big(s(p), g(p)\big), \qquad \text{for all } g \in V \text{ and } p \in F.$$

A similar formula holds for φ, and we deduce that

$$(\bar{\varphi} \circ \varphi) g(p) = g(p), \qquad \text{for all } g \in V \text{ and } p \in F.$$

Since, for each $g \in V$, both g and $(\bar{\varphi} \circ \varphi)g$ are continuous and affine, the barycenter formula holds for both, and we conclude that the two functions agree on all of K, and hence represent the same element of V. Similar reasoning shows that $(\varphi \circ \bar{\varphi})g$ always equals g. We have shown that the maps $\bar{\varphi}$ and φ are inverse to one another. It is clear that both φ and $\bar{\varphi}$ are order preserving.

Let $f \in X$. Since X is directed downward, there is an element g of X such that $g \leq 0$ and $g \leq f$. Note that g is a fixed point of φ. It follows that $g \leq \varphi f$, and we deduce that φf is in X, since this set is upper. This establishes that φ leaves the set X invariant. A similar argument shows that the same is true for $\varphi^{-1} = \bar{\varphi}$. The restriction of φ to X is the order isomorphism we require.

It remains to show that this restriction is not affine. This follows since $\varphi g(w) = \Gamma(1, g(w))$ is not affine in g. ☐

4. Order ideals in cones

Our goal now is to study order isomorphisms between subsets of complete order unit spaces. Since initially we have no other information about such maps than that they preserve the order structure, it will necessary to start by considering some purely order-theoretic notions.

Let P be a partially ordered set. This means that P is equipped with a partial order, in other words, a binary relation $\leq$ that is reflexive, antisymmetric, and transitive. A subset U of P is said to be *lower* if $x \in P$ is in U whenever $y \in U$ and $x \leq y$. It is said to be *directed upward* if, for every x and y in U, there exists z in U such that $x \leq z$ and $y \leq z$. The subset U is an *ideal* if it is non-empty, lower, and directed upward. An ideal in a partially ordered set P is called *proper* if it does not equal the whole of P.

Let (V, C, u) be a complete order-unit space. We denote by $\mathcal{I}$ the set of all proper ideals of the cone C. Our strategy will be to study an order isomorphism between the cones of order-unit spaces by examining its behaviour on the ideals of the cones. In particular, we are interested in ideals such as $\{x \in C \mid \langle y, x\rangle < \lambda\}$, where y is a pure state and $\lambda > 0$. However, in general, the ideals of C can be quite complicated. Note the result of Namioka and Saeki [19] who proved that, for some compact spaces X, there are order isomorphisms on the cone $C(X)^+$ of non-negative real-valued functions on X that do not preserve the strictly-less-than relation. Consequently, this strict relation cannot be defined in terms of the relation $\leq$. So, the ideals $\{x \in C \mid \langle y, x\rangle < \lambda\}$ are not necessarily mapped to similar such ideals.

For every ideal $I \in \mathcal{I}$, we define $\sup I$ to be the pointwise supremum of the elements of I, considered as functions on the state space K. The function $\sup I$ is non-negative and lower semicontinuous on K. Less obviously, it is also affine. This is because the map $\alpha \mapsto \alpha$ on I is a non-decreasing net of functions, and hence converges pointwise, necessarily to an affine function since each term is affine. Another fact that we will need is that the supremum of any proper ideal is not identically infinity.

For each pure state y and $\lambda \geq 0$, we denote by $L_{y,\lambda}$ the function on K that takes the value λ at y, and the value infinity everywhere else. We also define $\mathcal{I}_{y,\lambda}$ to be the set of proper ideals I of C such that $\sup I = L_{y,\lambda}$.

Lemma 4.1. *For each pure state y and $\lambda > 0$, the set of ideals $\mathcal{I}_{y,\lambda}$ is non-empty.*

Proof. Let $I := \{x \in C \mid \langle y, x\rangle < \lambda\}$. This set is clearly lower, and is non-empty because it contains 0. It is also directed upward, since $L_{y,\lambda}$ is a lower semicontinuous affine function on K—see the proof of Corollary I.1.4 of Alfsen [3]. Again from the same corollary, there is a non-decreasing net of elements of I converging pointwise on K to $L_{y,\lambda}$. Therefore, $\sup I$ is greater than or equal to $L_{y,\lambda}$. The opposite inequality is obvious. □

Lemma 4.2. *Let I_1 and I_2 be ideals of C such that $\sup I_1(y) < \sup I_2(y)$, for all $y \in K$ such that $I_1(y)$ is finite. Then $I_1 \subset I_2$.*

Proof. Given $x \in I_1$, we can approximate $\sup I_2$ closely enough with an element of I_2, so that the element is above x. So, x is in I_2. □

The set containment relation $\subset$ is a partial order on $\mathcal{I}$. It turns out that we can characterise the elements of $\mathcal{I}$ that are of the form $\mathcal{I}_{y,\lambda}$, with y a pure state and $\lambda \geq 0$, using only this partial order. It even allows us, given two ideals of this form, to tell whether the associated pure states are the same. This is the content of the next two lemmas. For each ideal $I \in \mathcal{I}$, denote by $I^{\uparrow}$ the set of ideals in $\mathcal{I}$ that contain I.

Lemma 4.3. *A proper ideal I is in $\mathcal{I}_{y,\lambda}$ for some pure state and $\lambda \geq 0$ if and only if $I^{\uparrow}$ is directed upward under the containment relation.*

Proof. Assume that I is in $\mathcal{I}_{y,\lambda}$, with y a pure state and $\lambda \geq 0$, and let I_1 and I_2 be proper ideals containing I. So, both $\sup I_1$ and $\sup I_2$ are greater than or equal to $\sup I$. It follows that $I_1 \in \mathcal{I}_{y,\lambda_1}$ and $I_2 \in \mathcal{I}_{y,\lambda_2}$, where λ_1 and λ_2 are non-negative real numbers greater than or equal to λ. Choose λ' strictly greater than both λ_1 and λ_2. By Lemma 4.1, $\mathcal{I}_{y,\lambda'}$ contains a proper ideal, and by Lemma 4.2 this ideal contains both I_1 and I_2. We have shown that $I^\uparrow$ is directed upward.

Now assume that I is a proper ideal such that $I^\uparrow$ is directed upward. Let y_1 and y_2 be pure states of K where $\sup I$ takes non-negative finite values λ_1 and λ_2, respectively. Take $\lambda_1' > \lambda_1$ and $\lambda_2' > \lambda_2$. By Lemma 4.1, there exist proper ideals $I_1 \in \mathcal{I}_{y_1,\lambda_1'}$ and $I_2 \in \mathcal{I}_{y_2,\lambda_2'}$. Both of these ideals contain I, by Lemma 4.2. Since $I^\uparrow$ is directed upward, it has as an element a proper ideal containing both I_1 and I_2. This is only possible if $y_1 = y_2$. We have shown that $\sup I$ is finite at only a single pure state, and since it is lower semicontinuous and affine it follows that it takes the form $L_{y,\lambda}$, with $y = y_1 = y_2$ and $\lambda = \lambda_1 = \lambda_2$. We deduce that I is in $\mathcal{I}_{y,\lambda}$. □

Lemma 4.4. *Let $I_1 \in \mathcal{I}_{y_1,\lambda_1}$ and $I_2 \in \mathcal{I}_{y_2,\lambda_2}$, where y_1 and y_2 are pure states, and λ_1 and λ_2 are non-negative real numbers. Then, $y_1 = y_2$ if and only if there is an element of $\mathcal{I}$ containing both I_1 and I_2.*

Proof. Assume that $y_1 = y_2 =: y$. Take a positive real number λ strictly greater than both λ_1 and λ_2. By Lemma 4.1 there is an ideal in $\mathcal{I}_{y,\lambda}$, and by Lemma 4.2 this ideal contains both I_1 and I_2.

Now assume that y_1 and y_2 differ, and that there is an ideal I containing both I_1 and I_2. So, $\sup I$ is greater than or equal to both $\sup I_1 = L_{y_1,\lambda_1}$ and $\sup I_2 = L_{y_2,\lambda_2}$, and hence takes the value ∞ everywhere on the state space K. This shows that I is the whole of C, and so not proper. □

The following lemma gives two criteria, one necessary and the other sufficient, for an element of C to be contained in any of the ideals in $\mathcal{I}_{y,\lambda}$.

Lemma 4.5. *Let g be a non-negative continuous affine real-valued function on K, and let I be an ideal in $\mathcal{I}_{y,\lambda}$, with y a pure state, and λ a non-negative real number. If $g(y) < \lambda$, then $g \in I$. If $g(y) > \lambda$, then $g \notin I$.*

Proof. Assume that $g(y) < \lambda$. The ideal I is directed upward, and so the map $\alpha \mapsto \alpha - g$ on this set is a net in V. Observe that this net is non-decreasing. Hence it converges pointwise to its supremum $L_{y,\lambda} - g$. Moreover, the infimum of $\alpha - g$ over K converges to the infimum of $L_{y,\lambda} - g$, which is $\lambda - g(y) > 0$; see Lemma 2.4 of [26]. So, we may choose $\alpha \in I$ large enough that $\alpha - g$ is positive on all of K, in other words, $\alpha > g$. Since I is lower, we deduce that $g \in I$. This proves the first part of the lemma. The second part follows directly from the definition of $\mathcal{I}_{y,\lambda}$. □

5. The behaviour of order ideals under order isomorphisms

In this section we study the behaviour of order isomorphisms by examining what they do to the ideals considered in the previous section.

We assume that we have an order isomorphism φ between the cones C and C' of order unit spaces V and V', respectively. Both spaces are assumed to be complete under their order unit norms. Recall that we are using F to denote the weak* closure of the set of extreme points of the state space K of the cone C. Observe that F is compact. We use similar notation concerning the cone C'.

Lemma 5.1. *There exists a bijection Φ between the pure states of V and those of V' such that if I is an ideal in $\mathcal{I}_{y,\lambda}$, with y a pure state and λ a non-negative real number, then φI is an ideal in $\mathcal{I}_{\Phi y,\lambda'}$, where λ' is some non-negative real number.*

Proof. Lemma 4.3 characterises the proper ideals having the property that they lie in $\mathcal{I}_{y,\lambda}$, for some pure state y and $\lambda \geq 0$. This characterisation is purely in terms of the order structure on C. A similar characterisation holds for the cone C'. It follows that a proper ideal is in $\mathcal{I}_{y,\lambda}$, with y an pure state of V and $\lambda \geq 0$, if and only if its image is in $\mathcal{I}_{y',\lambda'}$ for some pure state y' of V', and some $\lambda' \geq 0$.

Combining this with Lemma 4.4, we get that if two ideals I_1 and I_2 lie in $\mathcal{I}_{y,\lambda_1}$ and $\mathcal{I}_{y,\lambda_2}$, respectively, then there exists a pure state y' of V' such that their images lie in $\mathcal{I}_{y',\lambda'_1}$ and $\mathcal{I}_{y',\lambda'_2}$, respectively, where λ'_1 and λ'_2 are non-negative real numbers. We define the map Φ so that $\Phi y := y'$. We do this for each pure state y of V.

Applying the same considerations to the inverse map φ^{-1}, we see that Φ is a bijection from the pure states of V to those of V'. □

Lemma 5.2. *Let x_α be a net of pure states of V converging to a point x of F, and assume that Φx_α converges to a point z of F'. If g_1 and g_2 in C are such that $g_1(x) < g_2(x)$, then $\varphi g_1(z) \leq \varphi g_2(z)$.*

Proof. Choose a positive number λ such that $g_1(x) < \lambda < g_2(x)$. By Lemma 4.1, we can find a net of proper ideals I_α, defined on the same directed set as x_α, such that $\sup I_\alpha = L_{x_\alpha,\lambda}$ for all α. Note that, for α large enough, $g_1(x_\alpha) < \lambda < g_2(x_\alpha)$, and so, by Lemma 4.5, I_α contains g_1 but does not contain g_2. So, for α large enough, the ideal φI_α contains φg_1 but not φg_2. By Lemma 5.1, for each α, the ideal φI_α is an element of $\mathcal{I}_{\Phi x_\alpha,\lambda'_\alpha}$, with $\lambda'_\alpha \geq 0$. We deduce using Lemma 4.5 that

$$\varphi g_1(\Phi x_\alpha) \leq \lambda'_\alpha \leq \varphi g_2(\Phi x_\alpha),$$

for α large enough. Taking the limit as α tends to infinity, we get the result. □

A set Z of functions defined on a set X is said to *separate* points if, for every pair of distinct points x and y in X, there exists a function f in Z such that $f(x)$ differs from $f(y)$.

Lemma 5.3. *If x_α is a net of pure states of V that converges to a limit in F, then $\Phi(x_\alpha)$ converges to a limit in F'.*

Proof. Denote the limit of x_α by x. Since F' is compact, it suffices to show that $\Phi(x_\alpha)$ has a most one limit point. Assume for the sake of contradiction that there are two distinct limit points y and z, and take subnets y_β and z_γ of x_α such that $\Phi(y_\beta)$ and $\Phi(z_\gamma)$ converge to y and z, respectively. The set C' separates K', and so there exists an element h of C' that takes distinct values on y and z. Switch y and z if necessary and choose $\lambda > 0$ such that $h(y) < \lambda < h(z)$. For $\rho > 0$ large enough,

$$g_1 := h + \rho u' \qquad \text{and} \qquad g_2 := (2\lambda + \rho)u' - h$$

are in C', where u' is the order unit of V'. Observe that $g_1(y) < g_2(y)$ and $g_1(z) > g_2(z)$. We apply Lemma 5.2 to the map φ^{-1} twice, once for each of these inequalities. This gives that $\varphi^{-1}g_1(x) = \varphi^{-1}g_2(x)$.

Let

$$\delta := \frac{g_2(y) - g_1(y)}{2} > 0.$$

The function $g_3 := g_1 + \delta u'$ is in C' and satisfies the relations $g_3(y) < g_2(y)$ and $g_3(z) > g_2(z)$. Using the same reasoning as before, we get that $\varphi^{-1}g_3$ and $\varphi^{-1}g_2$ agree at x.

Let $g_4 := g_2 + \delta u'$. Again, $\varphi^{-1}g_3$ and $\varphi^{-1}g_4$ agree at x. Continuing in this manner, we see that the functions $\varphi^{-1}(g_1 + n\delta u')$ all take the same value at x, independently of $n \in \mathbb{N}$. But every element of C' lies below $g_1 + n\delta u'$ for some value of $n \in \mathbb{N}$, and φ^{-1} is an order isomorphism between C' and C. It follows that no element of C takes a value strictly greater than $\varphi^{-1}g_1(x)$ at x, which is absurd. □

A *regular* topological space is one in which each closed set can be separated by neighbourhoods from each point that it does not contain. More precisely, a topological space is regular if, given a closed set G and any point x not in G, there exist a neighbourhood of x and a neighbourhood of G that are disjoint. Every compact Hausdorff space is regular.

Recall the following theorem from [8, §8.5, p.81]. Let $f\colon A \to Y$ be a mapping from a dense subset A of a topological space X into a regular Hausdorff space Y. Then, f can be extended continuously to the whole of X if and only if, for each $x \in X$, we have that $f(y)$ converges in Y as y tends to x while remaining in A. The continuous extension is of course unique, if it exists.

Proposition 5.4. *The mapping Φ extends in a unique way to a homeomorphism between F and F'. This extension, which we again denote by Φ, has the property that, if $g_1(x) < g_2(x)$ for some $x \in F$ and g_1 and g_2 in C, then $\varphi g_1(\Phi x) \le \varphi g_2(\Phi x)$.*

Proof. From Lemma 5.3 and the theorem from [8] mentioned above, we get that Φ extends in a unique way to a continuous mapping between F and F'. Similar reasoning shows that the inverse mapping Φ^{-1} also extends to a continuous mapping, this time between F' and F, and it is easy to see that this extension is the inverse of the extension of Φ. Thus, the extension of Φ is a homeomorphism.

The second part is proved by applying Lemma 5.2 to any net of pure states of V converging to x. □

We will need to know how the homeomorphism induced by a product of order isomorphisms is related to those induced by the factors.

Lemma 5.5. *Let C, C', and C'' be the cones of complete order-unit spaces, and let $\varphi\colon C \to C'$ and $\varphi'\colon C' \to C''$ be order isomorphisms. Let $\Phi\colon F \to F'$ and $\Phi'\colon F' \to F''$, respectively, be the homeomorphisms they induce, and let $\Phi''\colon F \to F''$ be the homeomorphism induced by $\varphi' \circ \varphi$. Then, $\Phi'' = \Phi' \circ \Phi$.*

Proof. Let x be a pure state of V, and choose $\lambda > 0$. By Lemma 4.1, $\mathcal{I}_{x,\lambda}$ contains an ideal I. Applying Lemma 5.1 to the maps φ and φ', we get that $\varphi' \circ \varphi(I)$ is in $\mathcal{I}_{(\Phi' \circ \Phi)x,\lambda'}$, for some $\lambda' \ge 0$. Applying the same lemma to the map $\varphi' \circ \varphi$ gives that $\varphi' \circ \varphi(I)$ is also in $\mathcal{I}_{\Phi'' x,\lambda''}$, for some $\lambda'' \ge 0$. This implies that $\Phi' \circ \Phi$ and Φ'' agree at x, and hence on the set of pure states of V, since x was arbitrary. Finally, we extend the conclusion to the whole of F using Proposition 5.4. □

For each $p \in F$ and $x \in \mathbb{R}_+$, define

$$\Lambda_p(x) := \varphi(xu)(q), \tag{2}$$

where $q := \Phi(p)$. We call these functions the *coordinate transformations* of φ because they govern how the q-coordinate of $\varphi(f)$ depends on the p-coordinate of $f \in C$, as can be seen from the following proposition.

Proposition 5.6. *Let $\varphi\colon C \to C'$ be an order isomorphism between the cones of two complete order-unit spaces V and V'. Denote by Φ the induced homeomorphism between F and F'. Let $p \in F$, and write $q := \Phi(p)$. Let $f \in C$. If $f(p)$ is positive and Λ_p is continuous at $f(p)$, then*

$$\varphi(f)(q) = \Lambda_p(f(p)).$$

Proof. Write $c := f(p) > 0$. Take $\varepsilon \in (0, c)$, and let $g_1 := (c - \varepsilon)u$ and $g_2 := (c + \varepsilon)u$. Observe that $g_1(p) < f(p) < g_2(p)$. So, by Proposition 5.4, $\varphi g_1(q) \le \varphi f(q) \le \varphi g_2(q)$, which is equivalent to $\Lambda_p(c-\varepsilon) \le \varphi f(q) \le \Lambda_p(c+\varepsilon)$. Letting ε approach zero and using the continuity of Λ_p gives the result. □

6. The case when the induced homeomorphism is the identity map

We consider first the special case of an order isomorphism having as domain and image the cone of the same complete order-unit space, and such that the coordinate homeomorphism induced by the order isomorphism is the identity map. The proof in the general case will later rely on these results.

Let $\varphi\colon C \to C$ be an order isomorphism on the cone C of a complete order-unit space (V, C, u).

6.1. **Smoothing.** Our method will involve differentiating the order isomorphism. To ensure that this is possible, we will first smooth it by convoluting it with a sufficiently smooth function.

Let $\Xi\colon \mathbb{R} \to \mathbb{R}_+$ be a function of differentiability class C^2, whose support is $[0, 1]$, and whose integral over this interval is equal to 1. Fix $\alpha > 0$. Recall that we are denoting by u the order unit of the space V. Define the smoothed map $\bar\varphi$ using the following Pettis integral:

$$\bar\varphi(f) := \int_0^1 \varphi\big(f + \alpha t u\big)\Xi(t)\,\mathrm{d}t, \qquad \text{for } f \in C. \tag{3}$$

See Section 2 for a definition of the Pettis integral. To show that this integral is well-defined, we use that the first factor of the integrand is isotone in t.

Lemma 6.1. *Let $f \in C$ and $\alpha > 0$, and define the function $V^* \to \mathbb{R}$, $y \mapsto I(y)$,*

$$I(y) := \int_0^1 \langle y, \varphi(f + \alpha t u)\rangle \Xi(t)\,dt.$$

This integral is well-defined, for each $y \in V^*$. The function I is continuous on the dual cone C^* in the weak topology.*

Proof. For each y in V^* define the function

$$h_y\colon [0,1] \to \mathbb{R}, \qquad t \mapsto \langle y, \varphi(f + \alpha t u)\rangle.$$

Observe that, when $y \in C^*$, the function h_y is isotone. Recall that every isotone real-valued function on a closed interval is Lebesgue integrable, and that the product of two integrable functions is integrable, provided that one of them is bounded. It follows that $h_y(\cdot)\Xi(\cdot)$ is integrable when $y \in C^*$. The same must then be true for all $y \in V^*$, because C^* generates V^*, in the sense that $V^* = C^* - C^*$. Hence the integral in the statement of the lemma is well-defined.

Let y_β be a net in C^* converging in the weak* topology to a point $y \in C^*$. Take any $\varepsilon > 0$. For $x \in [0,1]$, define

$$\sigma(x) := \int_0^x \Xi(t)\,\mathrm{d}t.$$

Note that $\sigma(0) = 0$ and $\sigma(1) = 1$. Partition $[0,1]$ into a finite number of intervals $[x_i, x_{i+1}]$, such that $\sigma(x_{i+1}) - \sigma(x_i) < \varepsilon$, for each i.

Observe that h_{y_β} converges pointwise to h_y, as β tends to infinity. So, for β large enough, $h_{y_\beta}(x_i)$ is within ε of $h_y(x_i)$, for all i. Using this and that h_y and all the h_{y_β} are non-decreasing, we get

$$\sum_i h_y(x_i)\big(\sigma(x_{i+1}) - \sigma(x_i)\big) \le I(y) \le \sum_i h_y(x_{i+1})\big(\sigma(x_{i+1}) - \sigma(x_i)\big) \qquad \text{and}$$

$$-\varepsilon + \sum_i h_y(x_i)\big(\sigma(x_{i+1}) - \sigma(x_i)\big) \le I(y_\beta) \le \sum_i h_y(x_{i+1})\big(\sigma(x_{i+1}) - \sigma(x_i)\big) + \varepsilon.$$

So,

$$\begin{aligned} |I(y) - I(y_\beta)| &\le \sum_i \big(h_y(x_{i+1}) - h_y(x_i)\big)\big(\sigma(x_{i+1}) - \sigma(x_i)\big) + \varepsilon \\ &< (M+1)\varepsilon, \end{aligned}$$

where $M := h_y(1) - h_y(0)$. We conclude that $I(y_\beta)$ converges to $I(y)$. $\square$

Lemma 6.2. *The Pettis integral in (3) is well-defined.*

Proof. It was shown in Lemma 6.1 that the function from $[0,1]$ to $\mathbb{R}$ defined by

$$t \mapsto \langle y, \varphi(f + \alpha t u)\rangle \Xi(t)$$

is integrable, for each y in V^*, and moreover that the function I in the statement of that lemma is continuous on C^*. It is clear that I is a linear functional on V^*.

Since I is weak* continuous and affine on K, and V is complete, there is an element of v of V such that $\langle y, v\rangle = I(y)$, for all $y \in K$. This equation extends to all y in V^*, since both sides are linear in y. $\square$

The smoothed map $\bar\varphi$ is an order-preserving map from C to itself, but is not necessarily bijective. As in the case of φ itself, the map $\bar\varphi$ may be described in terms of how it transforms coordinates. The smoothing however gives us stronger properties of these transformations. For each $p \in F$, define the following function from $\mathbb{R}_+$ to $\mathbb{R}_+$:

$$\bar\Lambda_p(x) := \int_0^1 \Lambda_p\big(x + \alpha t\big)\Xi(t)\,\mathrm{d}t, \qquad \text{for all } x \in \mathbb{R}_+. \tag{4}$$

This function is obviously non-decreasing.

Lemma 6.3. *For each $p \in F$, the map $\bar\Lambda_p$ satisfies*

$$\big(\bar\varphi f\big)(p) = \bar\Lambda_p\big(f(p)\big), \qquad \textit{for all } f \in C. \tag{5}$$

Moreover, each of the maps $\bar\Lambda_p$ is differentiable on $(0,\infty)$.

Proof. Fix $p \in F$ and $f \in C$. By Proposition 5.6, for all except a countable number of values of $t \in \mathbb{R}_+$,

$$\varphi\big(f + \alpha t u\big)(p) = \Lambda_p\big(f(p) + \alpha t\big).$$

Multiplying by $\Xi(t)$ and integrating from 0 to 1, we get (5).

Next, we calculate the derivative of $\bar{\Lambda}_p$. Fix $x > 0$, and let $\delta \in \mathbb{R}$ be sufficiently close to zero that $x + \delta > 0$. We have

$$\bar{\Lambda}_p(x+\delta) = \int_{-\infty}^{\infty} \Lambda_p\big(x + \delta + \alpha t\big)\Xi(t)\,\mathrm{d}t.$$

Note that we have extended the range of integration to the whole of the real line. We may do this even though Λ_p is undefined for negative arguments because Ξ takes the value zero whenever such a negative argument occurs. For clarity, we use the convention that Λ_p takes the value zero on the negative halfline.

Make the change of variables $\tau := t + \delta/\alpha$ to obtain

$$\bar{\Lambda}_p(x+\delta) = \int_{-\infty}^{\infty} \Lambda_p\big(x + \alpha\tau\big)\Xi\Big(\tau - \frac{\delta}{\alpha}\Big)\,\mathrm{d}\tau. \tag{6}$$

Subtracting $\bar{\Lambda}_p(x)$, dividing by δ, and taking the limit as δ tends to zero gives, by the dominated convergence theorem,

$$\bar{\Lambda}'_p(x) = -\frac{1}{\alpha}\int_{-\infty}^{\infty} \Lambda_p\big(x + \alpha\tau\big)\Xi'(\tau)\,\mathrm{d}\tau. \tag{7}$$

We conclude that $\bar{\Lambda}_p$ is differentiable. □

6.2. **The Gateaux derivative.** We will consider the Gateaux derivative of the smoothed map $\bar{\varphi}$. This is defined to be, for $h \in \operatorname{int} C$ and $f \in V$,

$$d\bar{\varphi}(h; f) := \lim_{\tau\to 0} \frac{\bar{\varphi}(h + \tau f) - \bar{\varphi}(h)}{\tau} = \frac{d}{d\tau}\bar{\varphi}(h + \tau f)\Big|_{\tau=0},$$

provided the limit exists.

Recall that Taylor's Theorem [4, p.604] implies the following. Let f be a function that is twice differentiable on an interval $I \subset \mathbb{R}$, and assume that $|f''(x)| \le M$ on I. Then, $f(x) = f(a) + f'(a)(x-a) + R(x, a)$, for all x and a in I, where the remainder term satisfies

$$|R(x,a)| \le M(x-a)^2/2, \qquad \text{for all } x \text{ and } a \text{ in } I.$$

Define D to be the set of functions g on F such that there exists a point $h \in \operatorname{int} C$ with

$$g(p) = \frac{d}{dt}\bar{\Lambda}_p(t)\Big|_{t=h(p)}, = \bar{\Lambda}'_p(h(p)) \qquad \text{for all } p \in F. \tag{8}$$

Lemma 6.4. *Let $h \in \operatorname{int} C$ and $f \in V$, and let g be as in (8). Then, for all $p \in F$,*

$$\lim_{\delta\to 0} \frac{1}{\delta}\Big(\bar{\varphi}\big(h + \delta f\big)(p) - \bar{\varphi}\big(h\big)(p)\Big) = f(p)g(p),$$

and the convergence is uniform in p.

Proof. From Lemma 6.3, we get

$$\frac{d}{d\delta}\bar{\varphi}(h + \delta f)(p)\Big|_{\delta=0} = \bar{\Lambda}'_p\big(h(p)\big)f(p), \qquad \text{for all } p \in F,$$

which establishes the first part of the lemma.

It remains to show that the convergence is uniform. In the same way we established (7), one can show that the second derivative of $\bar{\Lambda}_p$ exists and is given by

$$\bar{\Lambda}''_p(x) = \frac{1}{\alpha^2}\int_{-\infty}^{\infty} \Lambda_p\big(x + \alpha\tau\big)\Xi''(\tau)\,\mathrm{d}\tau, \qquad \text{for all } p \in F \text{ and } x > 0.$$

Observe the following facts. Firstly, the second derivative Ξ'' is bounded on $[0,1]$ and is zero outside this interval. Secondly, Λ_p is non-decreasing, for each $p \in F$. Thirdly, for fixed $x > 0$, the map $p \mapsto \Lambda_p(x+\alpha)$ is continuous, and hence bounded. We conclude that $\bar{\Lambda}''_p$ is bounded uniformly in p on every compact interval of $(0,\infty)$.

Let I be a compact interval of $(0,\infty)$ containing $\{h(p) \mid p \in F\}$ in its interior. So, I also contains $\{h(p)+\delta f(p) \mid p \in F\}$, for δ small enough. Let $M > 0$ be such that $|\bar{\Lambda}''_p(x)| \le M$ for all $x \in I$ and $p \in F$. Taylor's theorem tells us that

$$\bar{\Lambda}_p(x) = \bar{\Lambda}_p(a) + \bar{\Lambda}'_p(a)(x-a) + R(x,a), \qquad \text{for all } x \text{ and } a \text{ in } I,$$

where the remainder term satisfies $|R(x,a)| \le M(x-a)^2/2$. Taking $a = h(p)$ and $x = h(p) + \delta f(p)$ gives

$$\Big|\frac{1}{\delta}\Big[\bar{\Lambda}_p\big(h(p)+\delta f(p)\big) - \bar{\Lambda}_p\big(h(p)\big)\Big] - \bar{\Lambda}'_p\big(h(p)\big)f(p)\Big| \le \frac{1}{2}M\delta f(p)^2,$$

for all $p \in F$, and δ small enough. The required uniform convergence now follows on applying Lemma 6.3. ☐

Lemma 6.5. *Let v_α be a net of points in V, such that $v_\alpha(\cdot)$ converges uniformly on F to a limit $f\colon F \to \mathbb{R}$. Then, v_α converges in the order-unit norm to a point v of V such that $v(\cdot) = f(\cdot)$ on F.*

Proof. For each $x \in K$, there exists a probability measure μ_x on F representing x. In particular,

$$v_\alpha(x) = \int_F v_\alpha \,\mathrm{d}\mu_x, \qquad \text{for all } \alpha.$$

It follows that v_α converges uniformly on K to some function, which is necessarily affine and continuous, and hence an element of V. ☐

Lemma 6.6. *Let $f \in V$ and $g \in D$. Then, there is a point v of V satisfying $v(p) = f(p)g(p)$, for all $p \in F$.*

Proof. Let h be the point in the open cone $\operatorname{int} C$ such that g satisfies (8). For all $\delta \in \mathbb{R}$ close enough to zero that $h + \delta f$ lies in C, let

$$h_\delta := \frac{1}{\delta}\big(\bar{\varphi}(h+\delta f) - \bar{\varphi}(h)\big).$$

By Lemma 6.4, h_δ converges uniformly on F, as δ tends to zero, to fg. The existence of a point v with the required property now follows upon applying Lemma 6.5. ☐

6.3. **A quotient of the pure state space.** We will define a quotient of the pure state space F by considering elements whose transformation functions are affine with the same slope to be equivalent. In other words, we define a relation $\sim$ on F, where $p \sim q$ if there are real numbers α, β, and λ such that $\bar{\Lambda}_p(x) = \alpha + \lambda x$ and $\bar{\Lambda}_q(x) = \beta + \lambda x$, for all $x > 0$. We denote by $F^\sim := F/{\sim}$ the topological quotient of F by this relation. Observe that every element of D is constant on each equivalence class of $\sim$, and hence may be considered to be a function on $F^\sim$. This function is moreover continuous. Indeed, taking f in Lemma 6.6 to be the unit, we see that every element of D agrees with some element of V on F, and is hence continuous on this set, which implies that it is continuous on the quotient.

Lemma 6.7. *Let p and q be points of F such that $p \not\sim q$. Then, there is a function in D that takes distinct values at p and q.*

Proof. If there exists $t \in (0, \infty)$ such that $\bar{\Lambda}'_p(t)$ and $\bar{\Lambda}'_q(t)$ differ, then the requirement of the lemma is satisfied by setting $h := tu$, where u is the unit, and defining an element of D according to (8). So, we assume that $\bar{\Lambda}'_p$ and $\bar{\Lambda}'_q$ agree on all of $(0, \infty)$. Combining this with the assumption of the lemma, we get that $\bar{\Lambda}'_p$ and $\bar{\Lambda}'_q$ are not constant.

Since $\operatorname{int} C$ separates F, it contains an element f such that $f(p)$ differs from $f(q)$. By switching labels if necessary, we may assume that $f(p) < f(q)$. Choose x and y in $(0, \infty)$, with $x < y$, such that $\bar{\Lambda}'_p(x) \neq \bar{\Lambda}'_q(y)$ and

$$\frac{y - x}{x} < \frac{f(q) - f(p)}{f(p)}.$$

Then, the point

$$h(\cdot) := \frac{y - x}{f(q) - f(p)} f(\cdot) + \Big(x - \frac{y - x}{f(q) - f(p)} f(p)\Big) u(\cdot)$$

is an element of $\operatorname{int} C$ since it is obtained from f by multiplying by a positive scalar and adding a positive multiple of the unit u. Observing that $h(p) = x$ and $h(q) = y$, we get the required element of D using (8). □

Lemma 6.8. *The quotient $F^{\sim}$ is compact and Hausdorff.*

Proof. The compactness follows immediately from the compactness of F.

Let p and q be distinct elements of $F^{\sim}$. By Lemma 6.7, there is an element g of D such that $g(p)$ and $g(q)$ are distinct. The map g is a continuous function on $F^{\sim}$. Take neighbourhoods in $\mathbb{R}$ of $g(p)$ and $g(q)$, respectively, that are disjoint from one another. The preimages of these neighbourhoods under g are disjoint neighbourhoods of p and q, respectively. This shows that $F^{\sim}$ is Hausdorff. □

The space $C(X, \mathbb{R})$ of continuous real-valued functions on a compact Hausdorff space X is an algebra of functions, that is, it is a linear space of functions that is closed under pointwise multiplication. Recall that a subalgebra of an algebra is a vector subspace that is closed under multiplication. The Stone–Weierstrass Theorem states the following. Assume X is a compact Hausdorff space and A is a subalgebra of $C(X, \mathbb{R})$ that contains a non-zero constant function. Then, A is uniformly dense in $C(X, \mathbb{R})$ if and only if it separates points of X. See [4, section 9.2] for more details.

Lemma 6.9. *Let μ be an affine dependency of the space V that is supported by F. Then, the pushforward μ_* of μ to $F^{\sim}$ by the quotient map is the zero measure.*

Proof. By Lemma 6.8, the quotient $F^{\sim}$ is a compact Hausdorff space. Observe that each element of D is constant on each equivalence class of F. So, such functions may be considered to be continuous functions on $F^{\sim}$.

Let P denote the set of functions that can be written as finite weighted sums of finite products of elements of D, that is, functions of the form

$$\sum_{i=1}^{n} c_i f^i_1 \cdots f^i_{m_i},$$

where $n \geq 1$, each $m_i \geq 0$, each f^i_j is in D, and each c_i is in $\mathbb{R}$. If m_i is zero for some i, then the associated term is a constant function. Note that P is a vector subspace of $C(F^{\sim}, \mathbb{R})$ that is closed under multiplication—in other words it is a subalgebra. Furthermore, it separates $F^{\sim}$ since it contains the set D, which is separating by Lemma 6.7. So, by the Stone–Weierstrass theorem, P is uniformly dense in $C(F^{\sim}, \mathbb{R})$.

Let $f \in P$. It follows from repeated application of Lemma 6.6 that there is an element of V that agrees with f on F. We denote this element again by f.

Since μ is an affine dependency, $\mu(f) = 0$. We deduce that its pushforward satisfies $\mu_*(f) = 0$, where f is now considered to be a function on the quotient space $F^\sim$. Since this is true for any f in a uniformly dense subset of $C(F^\sim, \mathbb{R})$, we see that μ_* is zero. □

6.4. **Linearity when the induced homeomorphism is the identity map.** We will need that the coordinate transformation functions of the smoothed map converge nearly everywhere to those of the original map, as the parameter α, which governs the scale over which the smoothing happens, tends to zero,

Lemma 6.10. *Let $p \in F$, and let $x \in (0, \infty)$ be such that Λ_p is continuous at x. Then, $\bar{\Lambda}_p(x)$ converges to $\Lambda_p(x)$, as α tends to zero.*

Proof. Since Λ_p is non-decreasing, $\Lambda_p(x) \le \Lambda_p(x + \alpha t) \le \Lambda_p(x + \alpha)$, for $t \in [0, 1]$. Using these bounds in (4), and that the area under Ξ in the interval $[0, 1]$ is 1, we get that $\Lambda_p(x) \le \bar{\Lambda}_p(x) \le \Lambda_p(x + \alpha)$, Letting α approach zero gives us the result. □

Lemma 6.11. *Let $x > 0$. The map $F \to \mathbb{R}_+$, $q \mapsto \bar{\Lambda}_q(x)$ is continuous.*

Proof. Taking $f = xu$ in Lemma 6.1, we get that $\bar{\varphi}(xu)(q)$ is continuous in q on F, since this set is a subset of C^*. By Lemma 6.3, this quantity equals $\bar{\Lambda}_q(x)$. □

Recall that we are assuming that Φ is the identity map.

Lemma 6.12. *If $p \in F$ is in the support of some affine dependency supported by F, then Λ_p is affine on $(0, \infty)$.*

Proof. Let $r \in F$ be such that Λ_r is not affine on $(0, \infty)$. Recall that, given any choice of $\alpha > 0$, we have defined in (3) a smoothed map $\bar{\varphi}$. By Lemma 6.10, $\bar{\Lambda}_r$ converges pointwise to Λ_r, as α tends to zero, everywhere on $(0, \infty)$ except at a countable number of points. If $\bar{\Lambda}_r$ were affine for all $\alpha > 0$, then this would imply that Λ_r was affine on all but a countable number of points of $(0, \infty)$, and hence on all of $(0, \infty)$, since Λ_r is non-decreasing. We conclude that there is some α for which $\bar{\Lambda}_r$ is not affine. Fix this α.

Take the quotient $F^\sim$ of F as above, by considering elements p and q to be equivalent if the maps $\bar{\Lambda}_p$ and $\bar{\Lambda}_q$ are affine with the same slope. Let μ be an affine dependency of the space V that is supported by F. By Lemma 6.9, its pushforward μ_* to $F^\sim$ is zero.

By Lemma 6.11, there is an open neighbourhood N of r such that $\bar{\Lambda}_p$ is not affine for any $p \in N$. Each element of N lies in an equivalence class consisting of a single point, and so the quotient map is a bijection between N and its image $N^\sim$. This implies that the restrictions to N of μ and μ_*, respectively, are equal. Hence, $|\mu|[N] = 0$. It follows that r is not in the support of μ. □

To go from the affineness of Λ_p to its homogeneity, we will introduce a new assumption. The following is the main result of this section.

Recall that we are considering a complete order-unit space V, and we are denoting by F its pure state space, and by L the closure of the union of the supports of the affine dependencies supported by F; see (1).

Lemma 6.13. *Let $\varphi \colon C \to C$ be an order isomorphism on the cone C of a complete order-unit space V. Assume that the homeomorphism Φ induced by φ on the pure state space is the identity map. Let $p \in L$ be such that*

$$\lim_{x \searrow 0} \Lambda_p(x) = 0. \tag{9}$$

Then, the coordinate transformation Λ_p *is homogeneous of degree* 1.

Proof. By assumption, there is a net p_β in F converging to p such that each p_β is in the support of an affine dependency supported by F. By Lemma 6.12, for each β, the map Λ_{p_β} is affine on $(0,\infty)$. Using that Λ_{p_β} converges pointwise to Λ_p, we get that Λ_p is affine on $(0,\infty)$. This, combined with (9), implies that it is homogeneous of degree 1. □

7. Homogeneity of the coordinate transformations

In this section, we return to considering a general order isomorphism φ between the cones C and C' of complete order-unit spaces V and V', respectively. We establish that each of the coordinate transformations Λ_p; $p \in L$ is homogeneous of some degree, possibly different from one. We will strengthen this conclusion later, with Lemma 9.1. Recall that F is the pure state space of V, and L is as defined in (1).

For each $\lambda > 0$, define the *homogeneity defect map*:

$$\Delta_\lambda \colon C \to C, \qquad \Delta_\lambda := \varphi^{-1} \circ M_\lambda \circ \varphi. \tag{10}$$

Here M_λ means multiplication by the scalar λ. The map Δ_λ measures how far away from being homogeneous the map φ is. Indeed, if φ is homogeneous, then Δ_λ is simply multiplication by λ. In any case, Δ_λ is an order isomorphism from C to itself. For each $\lambda > 0$ and $p \in F$, denote by $\Phi^{(\lambda)}$ the homeomorphism on F induced by Δ_λ, and denote by $\Lambda_p^{(\lambda)}$ the pth-coordinate transformation function as defined in (2). These maps are related to one another by Proposition 5.6.

The first lemma of this section shows that the main result of the previous section applies to the homogeneity defect map.

Lemma 7.1. *For all* $\lambda > 0$*, the homeomorphism* $\Phi^{(\lambda)}$ *on* F *induced by the defect map* Δ_λ *is the identity map. Moreover,* $\Lambda_p^{(\lambda)}$ *is homogeneous of degree* 1*, for each* $p \in L$.

Proof. The first part follows from Lemma 5.5.

To prove the second part, fix $p \in L$. Consider first the case where $0 < \lambda \le 1$. Observe that since $M_\lambda(g) \le g$, for all $g \in C'$, we have $\Delta_\lambda(g) \le g$, for all $g \in C$. We deduce that $\Lambda_p^{(\lambda)}(x) \le x$, for all $x \ge 0$. This implies that (9) holds. So, by Lemma 6.13, we get that $\Lambda_p^{(\lambda)}$ is homogeneous of degree 1.

When $\lambda \ge 1$, we apply the same argument to $\Delta_{1/\lambda}$, to get that $\Lambda_p^{(1/\lambda)}$ is homogeneous of degree 1, and then use that $\Delta_\lambda = (\Delta_{1/\lambda})^{-1}$, and so $\Lambda_p^{(\lambda)}$ is the inverse of $\Lambda_p^{(1/\lambda)}$, and hence homogeneous of degree 1. □

Lemma 7.2. *For each* $p \in L$*, the coordinate transformation* Λ_p *is homogeneous of some degree* $\alpha(p) > 0$.

Proof. Fix $p \in L$. For each $\lambda > 0$, define the homogeneity defect map as in (10). By Lemma 7.1, the associated coordinate transformation maps can be written

$$\Lambda_p^{(\lambda)}(x) = \rho(\lambda)x, \qquad \text{for all } \lambda > 0 \text{ and } x \ge 0,$$

where $\rho \colon (0,\infty) \to \mathbb{R}_+$ is some function.

Note that, for all positive λ and λ', we have $\Delta_\lambda \circ \Delta_{\lambda'} = \Delta_{\lambda\lambda'}$. The homeomorphism induced by each of Δ_λ, $\Delta_{\lambda'}$, and $\Delta_{\lambda\lambda'}$ is the identity map, and the coordinate transformations $\Lambda_p^{(\lambda)}$, $\Lambda_p^{(\lambda')}$, and $\Lambda_p^{(\lambda\lambda')}$ are all continuous on $(0,\infty)$. So, from Proposition 5.6 we get that $\Lambda_p^{(\lambda)} \circ \Lambda_p^{(\lambda')} = \Lambda_p^{(\lambda\lambda')}$. It follows that $\rho(\lambda)\rho(\lambda') = \rho(\lambda\lambda')$. The function ρ is non-decreasing on its domain $(0,\infty)$. Clearly ρ cannot be identically zero, because each

Δ_λ is an order isomorphism. According to [10, Theorem 1.49], the only other possibility is that $\rho(\lambda) = \lambda^c$, for all $\lambda > 0$, where c is a non-negative real number.

Observe that $\varphi \circ \Delta_\lambda = M_\lambda \circ \varphi$, for each $\lambda > 0$. We deduce that

$$\Lambda_p(\lambda^c x) = \lambda \Lambda_p(x), \qquad \text{for all } \lambda > 0 \text{ and } x \geq 0.$$

Recall that Λ_p is non-decreasing and unbounded, but not necessarily continuous. By evaluating both sides of this equation at a point where Λ_p is positive and continuous, taking $\lambda \neq 1$, we see that c cannot be zero.

Substituting $\gamma := \lambda^c$, we get

$$\Lambda_p(\gamma x) = \gamma^{1/c} \Lambda_p(x), \qquad \text{for all } \gamma > 0 \text{ and } x \geq 0.$$

This shows that Λ_p is homogeneous of degree $1/c$. □

8. Differentiating order isomorphisms

As before, we assume that (V, C, u) and (V', C', u') are complete order-unit spaces. Recall that F is the pure state space of V, and L is as defined in (1), and that analogous notation is used concerning V'.

The next lemma provides a way of obtaining a linear order-isomorphism between the spaces V and V', provided there is a sufficiently regular order isomorphism between their cones.

Lemma 8.1. *Let $\varphi\colon C \to C'$ be an order isomorphism, such that $\varphi(u) = u'$. Denote by Φ the homeomorphism it induces from F to F'. Assume that each coordinate transformation Λ_p; $p \in F$ is homogeneous of some degree $\alpha(p) > 0$. Then, φ is Gateaux differentiable at u, and its derivative in the direction $f \in V$ is given by*

$$d\varphi(u; f)(q) = \alpha(p) f(p), \qquad \textit{for all } q \in F',$$

where $p := \Phi^{-1}(q)$. The map $f \to d\varphi(u; f)$ is a linear order-isomorphism from V to V', and it induces the same homeomorphism between F and F' as φ, namely Φ.

Proof. Fix $f \in V$. For each $p \in F$, we have that $\Lambda_p(1) = 1$, and so $\Lambda_p(x) = x^{\alpha(p)}$, for all $x \in \mathbb{R}_+$. Since $\varphi(2u)$ is continuous on F', and $\varphi(2u)(q) = 2^{\alpha(p)}$, for each $q \in F'$, where $p = \Phi^{-1}(q)$, we see that α is continuous on F, and hence bounded. This implies that the second derivative of Λ_p is bounded uniformly in p in an interval neighbourhood $I \subset (0, \infty)$ of 1, that is, there exists $M > 0$ such that

$$\Big| \frac{d^2 \Lambda_p(x)}{dx^2} \Big| \leq M, \qquad \text{for all } p \in F \text{ and } x \in I.$$

Taylor's theorem gives that, for $p \in F$ and δ small enough,

$$\begin{aligned} \varphi(u + \delta f)(\Phi(p)) &= \Lambda_p\big((u + \delta f)(p)\big) \\ &= 1 + \delta \alpha(p) f(p) + R(\delta, p, f(p)), \end{aligned}$$

with the remainder term satisfying $|R(\delta, p, f(p))| \leq M \delta^2 f(p)^2 / 2$.

This shows that the limit in the definition of the Gateaux derivative converges uniformly on F' to $\alpha(p) f(p)$. By Lemma 6.5, this means that the Gateaux derivative $d\varphi(u; f)$ exists and has the required form.

The map $\mathcal{D}\colon f \mapsto d\varphi(u; f)$ is obviously linear and order preserving.

Consider the inverse φ^{-1} of φ. Each of the coordinate transformations Λ'_q, with $q \in F'$, of this map is also homogeneous, this time of degree $1/\alpha(p)$, where $p := \Phi^{-1}(q)$. Using the same reasoning as for φ, we get that the Gateaux derivative of φ^{-1} at the order unit u' in the direction $f \in V'$ is given by

$$d\varphi^{-1}(u'; f)(p) = \frac{1}{\alpha(p)} f(q), \qquad \text{for all } q \in F',$$

where $p := \Phi^{-1}(q)$. From this expression, we see that the map $f \mapsto d\varphi^{-1}(u'; f)$ is also linear and order preserving. Moreover, it is the inverse of the map $\mathcal{D}$. Hence, $\mathcal{D}$ is a linear order-isomorphism between V and V'.

It remains to show that the homeomorphism $\Phi^{\mathcal{D}}$ induced by $\mathcal{D}$ is the same as that induced by φ. Let p be a pure state of V, and choose $\lambda > 0$. By Lemma 4.1, the set $\mathcal{I}_{p,\lambda}$ contains an ideal I, satisfying $\sup I = L_{p,\lambda}$. Since $\mathcal{D}$ is an order isomorphism, it maps I to an ideal I' of C'. We see from the explicit formula for $\mathcal{D}$ that $\sup I'$ takes the value $\alpha(p)\lambda$ at $\Phi(p)$ and the value infinity everywhere else. So, I' is in $\mathcal{I}_{\Phi(p),\alpha(p)\lambda}$. It follows using Lemma 5.1 that $\Phi^{\mathcal{D}}(p) = \Phi(p)$. Since this holds for an arbitrary pure state of V, the pure states are dense in F, and $\Phi^{\mathcal{D}}$ and Φ are continuous, the conclusion follows. $\square$

A *normal* topological space is one in which every two disjoint closed sets have disjoint open neighborhoods. Every compact Hausdorff space is normal. Recall that *Tietze's extension theorem* states that every continuous real-valued function defined on a closed subset of a normal topological space can be extended to a continuous function on the whole space, and, moreover, that if the function is bounded, then the extended function can be chosen to have the same bounds.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality, we may assume that the domain of φ is C and the co-domain is C'—if this is not the case, then choose some element f of X and consider the map $g \mapsto \varphi(f + g) - \varphi(f)$ from C to C'. By Lemma 2.1, we can choose order units u and u' of V and V' respectively, such that $\varphi(u) = u'$.

Recall that we are using L to denote the closure of the union of the supports of the affine dependencies of V supported by F. By Lemma 7.2, for each $p \in L$, the coordinate transformation Λ_p is homogeneous of some positive degree. Likewise, the coordinate transformation Λ'_q of the inverse map φ^{-1} is homogeneous of positive degree, for each $q \in L'$, where L' is defined analogously to L. Observe that Λ_p and Λ'_q are inverses when $q = \Phi(p)$. It follows that Λ_p is homogeneous of positive degree for all $p \in \Phi^{-1}(L')$. Therefore, we can write

$$\Lambda_p(x) = x^{\alpha(p)}, \qquad \text{for all } p \in L \cup \Phi^{-1}(L') \text{ and } x \in \mathbb{R}_+,$$

where α is a positive function.

By considering the image under φ of $2u$, where u is the unit, we see that α is continuous on $L \cup \Phi^{-1}(L')$. Since L and L' are compact, α is bounded above and below by positive numbers on this set.

By Tietze's extension theorem, we may extend α to a continuous function on the whole of F. We reuse the symbol α to denote this extended map. The extension can be chosen to be bounded above and below by positive numbers.

Let $h \in C$, and consider the function $f : F' \to \mathbb{R}$ defined by

$$f(q) := h(p)^{\alpha(p)}, \qquad \text{for all } q \in F',$$

where p and q are related by $q = \Phi(p)$. The function f is continuous. Moreover, it agrees with $\varphi(h)$ on L', and so $\mu(f) = 0$, for each affine dependency μ of V' supported by F'.

Applying Lemma 2.2, we get that there is a element g_h of V' such that $g_h(q) = f(q)$, for all $q \in F'$. Since g_h takes non-negative values on F', it is an element of C'.

Define the map $\kappa\colon C \to C'$ by setting $\kappa(h) := g_h$, for all $h \in C$. This map is clearly order preserving. Similar reasoning to that in the previous paragraph shows that one may define a map from C' to C having coordinate transformations

$$\Lambda'_q(x) = x^{1/\alpha(p)}, \qquad \text{for all } q \in F',$$

where $p = \Phi^{-1}(q)$. This map is the inverse of κ and is also order preserving. Hence κ is an order isomorphism. Observe that $\kappa(u) = u'$. It only remains to apply Lemma 8.1 to get that there is a linear order-isomorphism between V and V'. □

9. Affineness of order isomorphisms

In this section, we complete the proofs of Theorems 1.3 and 1.5.

First, we sharpen Lemma 7.2 to say that the degree of homogeneity of the coordinate transformation maps is actually 1, for elements of L.

Lemma 9.1. *Let $\varphi\colon C \to C'$ be an order isomorphism between the cones of two complete order-unit spaces V and V'. Then, for each $p \in L$, the coordinate transformation Λ_p is homogeneous of degree 1.*

Proof. By Theorem 1.1, there is a linear order-isomorphism $\mathcal{D}$ from V' to V. Moreover, this map induces the same homeomorphism between F' and F as φ^{-1}. By Lemma 5.5, this homeomorphism is the inverse of the one Φ induced by φ between F and F'.

Consider the map $\vartheta := \mathcal{D} \circ \varphi$. This is an order isomorphism from C to itself, and, by Lemma 5.5, the homeomorphism it induces on F is the identity map. By Lemma 7.2, each of the coordinate transformations Λ_p; $p \in L$ of φ is homogeneous of some positive degree. Furthermore, every coordinate transformation $\Lambda^{\mathcal{D}}_q$; $q \in F'$ of $\mathcal{D}$ is homogeneous of degree 1. Let Λ^{ϑ}_p; $p \in F$, be the coordinate transformations of ϑ. Observe that, for each $p \in L$, we have $\Lambda^{\vartheta}_p = \Lambda^{\mathcal{D}}_q \circ \Lambda_p$, where $q := \Phi(p)$, and hence the coordinate transformation Λ^{ϑ}_p is homogeneous of the same degree as Λ_p. This implies that (9) holds, and Lemma 6.13 is therefore applicable. We deduce that, for all $p \in L$, the transformation Λ^{ϑ}_p is homogeneous of degree 1, and hence so is Λ_p. □

The next lemma generalises our results about isomorphisms between cones to isomorphisms between upper sets.

Lemma 9.2. *Let V and V' be two complete order-unit spaces, and let $\varphi\colon U \to U'$ be an order isomorphism between non-empty upper sets $U \subset V$ and $U' \subset V'$. Then, there exists a homeomorphism Φ between F and F' such that, for each $p \in L$, the following holds: there exists an increasing affine function $a\colon \mathbb{R} \to \mathbb{R}$ satisfying $\varphi(f)(q) = a(f(p))$, for all $f \in U$, where $q := \Phi(p)$.*

Proof. Let $z \in U$. The map

$$\vartheta_z\colon C \to C', \quad f \mapsto \varphi(f + z) - \varphi(z)$$

is an order isomorphism from the cone C of V to the cone C' of V'. Let Φ be the homeomorphism from F to F' that this induces.

Let $p \in L$, and write $q := \Phi(p)$. From Lemma 9.1, the coordinate transformation Λ_p of ϑ_z is homogeneous of degree 1. This means that there is some $\lambda > 0$ such that $\vartheta_z(f)(q) = \lambda f(p)$, for all $f \in C$. Note that, for all $f \geq z$,

$$\varphi(f) = \vartheta_z(f - z) + \varphi(z),$$

and so,

$$\begin{aligned}\varphi(f)(q) &= \lambda f(p) - \lambda z(p) + \varphi(z)(q)\\ &= a(f(p)),\end{aligned}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is defined to be $a(x) := \lambda x + c$, with $c := -\lambda z(p) + \varphi(z)(q)$.

Now let w be another point of U, and define ϑ_w in a similar way to how we defined ϑ_z. Again, we get a homeomorphism from F to F', taking p to some element q' of F'. As before, we have $\varphi(f)(q') = a'(f(p))$, for all $f \geq w$, where a' is some increasing affine function from $\mathbb{R}$ to itself.

Since V is an order unit space, it has an element x that is greater than both z and w. Consider the set $X := \{f \in V \mid f \geq x\}$. Every point f of this set satisfies both $f \geq z$ and $f \geq w$. Its image is $\varphi(X) = \{g \in V' \mid \varphi(x) \leq g\}$. For g in $\varphi(X)$, both $g(q)$ and $g(q')$ only depend on $\varphi^{-1}(g)(p)$. By varying g within $\varphi(X)$, we see that q and q' are equal. Comparing the expressions for $\varphi(\cdot)(q)$ and $\varphi(\cdot)(q')$ on X then gives that the two affine functions a and a' are equal.

We have shown that the homeomorphism Φ and the affine function a do not depend on the choice of z. We conclude that the equation $\varphi(f)(q) = a(f(p))$ holds for all $f \in U$. □

We can now proof the other half of Theorem 1.3.

Lemma 9.3. *Let V and V' be two complete order-unit spaces, and let $\varphi\colon U \to U'$ be an order isomorphism between non-empty upper sets $U \subset V$ and $U' \subset V'$. Assume that V satisfies Condition 1.2. Then, φ is affine.*

Proof. Denote by Φ the homeomorphism from F to F' given by Lemma 9.2. Let f and g in U and $\alpha \in [0, 1]$ be such that $(1-\alpha)f + \alpha g$ lies in U.

Let $q \in F'$, and let $p \in F$ be such that $q = \Phi(p)$. By assumption, p is in L. So, by Lemma 9.2 again, there exists an increasing affine function $a\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(h)(q) = a(h(p))$, for all $h \in U$. Therefore,

$$\begin{aligned}\varphi\big((1-\alpha)f + \alpha g\big)(q) &= a\big((1-\alpha)f(p) + \alpha g(p)\big)\\ &= (1-\alpha)a\big(f(p)\big) + \alpha a\big(g(p)\big)\\ &= \big((1-\alpha)\varphi(f) + \alpha\varphi(g)\big)(q).\end{aligned}$$

Since this is true for all $q \in F'$, we deduce that

$$\varphi\big((1-\alpha)f + \alpha g\big) = \big((1-\alpha)\varphi(f) + \alpha\varphi(g)\big).$$

It follows that φ is an affine map. □

Finally, we can prove Theorems 1.3 and 1.5.

Proof of Theorem 1.3. If Condition 1.2 holds, then Lemma 9.3 shows that every order isomorphism between U and U' is affine.

Assume now that U is directed downward, that all order isomorphisms between U and U' are affine, and that there exists an order isomorphism ϑ between U and U', which is of course necessarily affine. If Condition 1.2 does not hold for the space V, then by Lemma 3.1 there exists an order isomorphism φ from U to itself that is not affine. In this case, the composition $\vartheta \circ \varphi$ is an order isomorphism between U and U' that is not affine. We conclude that Condition 1.2 holds for V. □

Proof of Theorem 1.5. Assume that the state space is not a Bauer simplex, and let φ be an order isomorphism between $\operatorname{int} C$ and a non-empty upper subset U of V. We must show that U does not equal V. Since the state space is not a Bauer simplex, there exists an element p of L. Let Φ be the homeomorphism on F given by Lemma 9.2, and write $q := \Phi(p)$. According to the same lemma, there is an increasing affine function $a\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(f)(q) = a(f(p))$, for all $f \in \operatorname{int} C$. Observe that $f(p)$ is positive for all $f \in \operatorname{int} C$. It follows that $\varphi(f)(q)$ is bounded below by $a(0)$ as f varies in $\operatorname{int} C$, which means that the map $g \mapsto g(q)$ is bounded below on U. Since this map is not bounded below on V, we deduce that U and V differ.

Now assume that the state space is a Bauer simplex, that is, that the set of affine dependencies supported by F is empty. Define the map $\varphi\colon \operatorname{int} C \to V$, $x \mapsto \varphi(x)$ by

$$\varphi(x)(p) := \log(x(p)), \qquad \text{for all } p \in F.$$

Since, for each $x \in \operatorname{int} C$, the function $\varphi(x)(\cdot)$ is continuous on F, we have by Lemma 2.2 that $\varphi(x)$ is a well-defined element of V. It is clear that φ is order preserving. Similar reasoning shows that the coordinate-wise exponential function defined on V is also well-defined and order preserving. Since this function is the inverse of φ, the proof is complete. □

10. Examples

In this section, we see how our results apply to some examples.

Example 10.1 (Finite dimensional cones)**.** We say that a finite-dimensional ordered vector space has no one-dimensional factors if it is not linearly order-isomorphic to a space of the form $V := V_1 \oplus \mathbb{R}$ with cone $C_1 \times [0, \infty)$, where (V, C_1) is an ordered vector space and $\oplus$ denotes the direct sum of two vector spaces. Note that a space has no one-dimensional factors if and only if the same is true for its dual.

Proposition 10.2. *A finite dimensional ordered vector space* (V, C) *satisfies condition 1.2 if and only if it has no one-dimensional factors.*

Proof. Let p be a pure state. If p is not isolated in the set of pure states, then it can be expressed as a linear combination of other pure states, and is thus in the support of an affine dependency supported by the pure state space. Indeed, this affine dependency consists of a finite number of atoms. On the other hand, if p is isolated and in the support of an affine dependency supported by the pure state space, then the dependency has an atom on p, which combined with Carathéodory's theorem [4, page 184] implies that p can again be written as a linear combination of other pure states. We deduce that every element of L is in the support of an atomic affine dependency supported by the pure state space. So, condition 1.2 is equivalent to every pure state being a linear combination of other pure states. This is equivalent to the dual space V^* having no one-dimensional factors, which in turn is equivalent to V having no one-dimensional factors. □

Example 10.3 (The Lorentz cone)**.** Consider, in particular, the space $\mathcal{L} := (\mathbb{R}^3, \Lambda, u)$, which is three-dimensional Euclidean space endowed with the Lorentz cone

$$\Lambda := \left\{(x, \lambda) \in \mathbb{R}^2 \times \mathbb{R} \mid ||x||_2 \leq \lambda\right\}$$

and the order unit $u := (0, 1)$. The state space here is the disk $K_{\mathcal{L}} := \{(x, 1) \mid ||x||_2 \leq 1\}$ in the dual space, and the set of pure states is its boundary circle $\partial_e K_{\mathcal{L}}$. If $p := (x_1, 1)$ and $q := (x_2, 1)$ are pure states such that x_1 and x_2 are linearly independent, then, writing $r := (-x_1, 1)$ and $s := (-x_2, 1)$, we have that $\delta_p - \delta_q + \delta_r - \delta_s$ is an affine dependency, whose support is composed of four points. Here, δ_x represents an atom of unit mass at a point x. Thus, every pure state appears in the support of some affine dependency, and so condition 1.2 holds.

Example 10.4 (The cone of non-negative continuous functions)**.** Let X be a compact Hausdorff space, and let $C(X)$ be the space of continuous real-valued functions on X. We use the pointwise order, which means that the relevant cone is the cone $C^+(X)$ of non-negative continuous functions. The dual space is the space of signed Radon measures on X. The states are the probability measures, and the pure states are the Dirac masses. There are no affine dependencies at all, and so Condition 1.2 does not hold.

Example 10.5 (Piece-of-string-in-a-cone)**.** Consider the space $C([0,1],\mathcal{L})$ of continuous maps from the interval $[0,1]$ to the three-dimensional space $\mathcal{L}$ ordered by the Lorentz cone; see example 10.3 above. The order we take is the pointwise order, and the cone is the set $C([0,1],\Lambda)$ of continuous maps into the Lorentz cone. Each point of this cone describes the configuration of an elastic string lying in the Lorentz cone.

The dual space is the space $M([0,1],\mathcal{L}^*)$ of regular Borel vector measures on $[0,1]$ with values in $\mathcal{L}^*$ that are of bounded variation; see [23, Lemma 1.6, p. 193]. Here, $\mathcal{L}^*$ denotes the dual space of $\mathcal{L}$. The set of pure states in this case can be identified with the Cartesian product $[0,1]\times\partial_e K_{\mathcal{L}}$, with the product topology. Here, $\partial_e K_{\mathcal{L}}$ is the set of pure states of the space $\mathcal{L}$. Each copy of $\partial_e K_{\mathcal{L}}$ has the same affine dependencies as $\partial_e K_{\mathcal{L}}$, and so $C([0,1],\mathcal{L})$ satisfies Condition 1.2.

An interesting variant of this example occurs when one of the endpoints of the string is constrained to lie on the axis of the Lorentz cone, that is, we consider the subspace of $C([0,1],\mathcal{L})$ consisting of those elements $x(\cdot)$ satisfying $x(0)\in\{0\}\times\mathbb{R}\subset\mathcal{L}$. The dual space is a quotient of the dual space of $C([0,1],\mathcal{L})$. In particular, the pure state space can be obtained from that of $C([0,1],\mathcal{L})$ be collapsing one end of the cylinder $[0,1]\times\partial_e K_{\mathcal{L}}$ to a point. There are no atomic affine dependencies with mass on this point; however, there are affine dependencies consisting of a countable number of atoms that have the collapsed point in their support.

The following example shows why it is necessary to consider the *closure* of the union of the supports of the affine dependencies, rather than just the union.

Example 10.6. Let ω_1 denote the first uncountable ordinal. The set $\Omega := \omega_1 + 1$ contains all the ordinals up to and including ω_1, and is compact when given its order topology; see [4, Section 2.37]. The space $C(\Omega,\mathcal{L})$ satisfies Condition 1.2 for the same reason that $C([0,1],\mathcal{L})$ does; see example 10.5.

Consider now the subspace Q of $C(\Omega,\mathcal{L})$ obtained by requiring the ω_1 coordinate to be on the axis of the Lorentz cone, that is,

$$Q := \big\{g\in C(\Omega,\mathcal{L}) \mid g(\omega_1)\in\{0\}\times\mathbb{R}\big\}.$$

Similarly to in the previous example, the set of pure states can be identified with the quotient of the product $\Omega\times\partial_e K_{\mathcal{L}}$ obtained when one copy of $\partial_e K_{\mathcal{L}}$, namely $\{\omega_1\}\times\partial_e K_{\mathcal{L}}$, is collapsed to a point, which we denote by $\overline{\omega}_1$. This quotient is a closed subset of the state space of Q. The point $\overline{\omega}_1$ is not isolated in the set of pure states, and each of the other pure states, that is, each point of $(\Omega\backslash\{\omega_1\})\times\partial_e K_{\mathcal{L}}$, is in the support of an affine dependency consisting of a finite number of atoms. We conclude that Q satisfies Condition 1.2.

However, $\overline{\omega}_1$ is not itself in the support of any affine dependency, as the following proposition shows.

Proposition 10.7. *The pure state $\overline{\omega}_1$ is not in the support of any affine dependency.*

Proof. Note that, if μ is a measure on Ω and $x \in \Omega$ is the least element of the support of μ, then $[0, x+1)$ is an open neighbourhood of x, and so $x = [0, x+1) \cap \operatorname{supp} \mu$ has positive mass, and hence there is an atom at x. It follows that any measure on Ω with no atoms must be zero, or stated another way, every measure on Ω consists entirely of atoms. The number of atoms, of course, must be countable.

Let μ be an affine dependency supported by the pure state space F of Q, and consider its positive and negative parts μ_+ and μ_-, respectively. Denote by μ_* the push forward of $|\mu| := \mu^- + \mu^+$ by the projection map $\pi \colon F \to \Omega$. Let M be the supremum of the set $\operatorname{supp} \mu_* \backslash \{\omega_1\}$. Note that M is a countable ordinal and so is strictly less than ω_1.

Let μ_+ and μ_- have atoms of mass m_+ and m_- at $\overline{\omega}_1$, respectively. The measures μ_+ and μ_- are mutually singular, and so at least one of m_+ and m_- is zero. We denote by $u := (0, 1)$ the order unit of the space $\mathcal{L}$. The function taking the value 0 on $[0, M]$ and the value u on $[M+1, \omega_1]$ is continuous, and so is an element of Q. It therefore gives rise to a continuous function on the pure state space. The integral of this function with respect to μ is $m_+ - m_-$, and so this quantity is zero. We conclude that μ does not have an atom at $\overline{\omega}_1$, and hence that μ_* does not have an atom at $\omega_1 = \pi(\overline{\omega}_1)$.

We have shown that $(M, \omega_1]$ does not intersect the support of μ_*. It follows that $\pi^{-1}(M, \omega_1]$ has measure zero with respect to the affine dependency μ. Since this set is an open set containing $\overline{\omega}_1$, we see that $\overline{\omega}_1$ is not in the support of μ. $\square$

Example 10.8 (Bounded operators on a Hilbert space)**.** Let $B(\mathcal{H})_{\mathrm{sa}}$ be the space of bounded self-adjoint linear operators on a Hilbert space $\mathcal{H}$. The cone in this case is the cone $B(\mathcal{H})_{\mathrm{sa}}^+$ of positive semi-definite elements of $B(\mathcal{H})_{\mathrm{sa}}$, that is, the elements A such that $\langle Ax, x\rangle \geq 0$, for all $x \in \mathcal{H}$. Recall that a *vector state* is a function of the form $A \mapsto \omega_x(A) := \langle Ax, x\rangle$, for $A \in B(\mathcal{H})_{\mathrm{sa}}$, where x is a unit vector in $\mathcal{H}$. It is known (see Theorem 4.3.9 and Exercises 4.6.68 and 4.6.69 of [9]) that every vector state is a pure state, and that the set of vector states is dense in the pure state space. Note however that when the dimension of $\mathcal{H}$ is infinite there exist pure states that are not vector states, and the set of pure states is not closed. Let x and y be linearly independent unit vectors in $\mathcal{H}$. Observe that the following algebraic relation holds:

$$\langle A(x+y), x+y\rangle + \langle A(x-y), x-y\rangle = 2\langle Ax, x\rangle + 2\langle Ay, y\rangle, \qquad \text{for } A \in B(\mathcal{H})_{\mathrm{sa}}.$$

Defining the unit vectors

$$w := \frac{x+y}{||x+y||} \qquad \text{and} \quad z := \frac{x-y}{||x-y||},$$

we see that the signed measure consisting of atoms of weight $||x+y||^2$ on ω_w and $||x-y||^2$ on ω_z, and atoms of weight -2 on ω_x and ω_y, is an affine dependency. We conclude that every vector state is in the support of an affine dependency, and, since these are dense in the pure state space, that condition 1.2 holds. This completes the proof of Corollary 1.4.

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